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DOMAIN DEPENDENCE OF CRITICAL RAYLEIGH NUMBERS

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Abstract—The dependence of critical Rayleigh numbers for buoyancy-driven convection on the size of the domain is investigated. The method of parameter differentiation is used to establish monotonicity conditions for the Rayleigh number–geometry dependence for axisymmetric convection in a cylindrical container and for three-dimensional convection in a rectangular box. The dependence of the critical Rayleigh number on the Nusselt number is also studied.

NOMENCLATURE

$\mathbf{d}$	gradient defined by equation (27)
$\mathbf{F}$	vector defined by equation (35)
g	magnitude of acceleration of gravity
h	heat transfer coefficient at free surface
k	thermal conductivity of liquid
L	matrix operator defined by equation (3)
L_1, L_2	dimensionless horizontal distance variables in x, y directions for rectangular box
l	height of cylindrical liquid layer
M	matrix operator defined by equation (28)
Nu	Nusselt number, hl/k
P_0	dimensionless modified pressure
P_0^*	$\partial P_0/\partial L_1$
$\mathbf{Q}$	vector defined by equation (26)
$\mathbf{Q}^*$	$\partial \mathbf{Q}/\partial L_1$
R	radius of cylindrical liquid layer
$(Ra)_0$	critical Rayleigh number defined by equation (15)
$(\overline{Ra})_0$	critical Rayleigh number defined by equation (16)
r	dimensionless radial distance variable
U_0, V_0, W_0	dimensionless velocity perturbation components for rectangular geometry
$\mathbf{v}_0$	dimensionless velocity perturbation vector for rectangular geometry
$\mathbf{v}_0^*$	$\partial \mathbf{v}_0/\partial L_1$
$\mathbf{w}_0$	vector defined by equation (2)
x, y, z	dimensionless coordinate variables for rectangular geometry
z	dimensionless axial distance variable for cylindrical geometry

Greek symbols

α	thermal coefficient of expansion
β	l/R
∇^2	Laplacian
θ_0	dimensionless temperature perturbation
κ	thermal diffusivity of liquid
ν	kinematic viscosity of liquid
ψ_0	dimensionless stream function perturbation

1. INTRODUCTION

ONE OF the more important aspects of the study of buoyancy-driven convection in bounded domains is the dependence of the critical Rayleigh number, $(Ra)_0$, on the size of the fluid container. In general, it is reasonable to expect higher values of $(Ra)_0$ when the lateral walls are closer together since the lateral boundaries can severely inhibit convective motion, thus introducing a stabilizing effect. Indeed, Ostrach and Pnueli [1] and Jennings and Sani [2] have proved that the critical Rayleigh number should be a nonincreasing function of the size of the domain for a fluid container with rigid, conducting walls. The calculations of Charlson and Sani [3] for a cylindrical container and of Jennings and Sani for a rectangular box are in accord with this prediction. However, the change of $(Ra)_0$ with the size of the domain is not necessarily monotonic when, for example, the side walls of the container are insulating rather than conducting. This has been illustrated by the calculations of Charlson and Sani for a cylindrical geometry and of Catton [4] for a rectangular geometry. In both investigations, local maxima were observed in plots of the critical Rayleigh number vs aspect ratio for the case of insulating sidewalls.

The general lack of monotonicity in the geometry dependence of the critical Rayleigh number complicates the interpolation of computed Rayleigh number–aspect ratio results. Consequently, it is of interest to see if some combination of $(Ra)_0$ and the aspect ratio does change monotonically, since the derivation of such a monotonicity condition will of course facilitate utilization of numerical results for the geometry dependence of the critical Rayleigh number. The purpose of this paper is to apply the technique of parameter differentiation to illustrate how a monotonicity result can be formulated for the $(Ra)_0$–aspect ratio dependence for a given geometry. Such a result can be derived for any particular eigenvalue of the linearized convection problem and, hence, necessarily holds for the principal eigenvalue, the critical Rayleigh number. We also illustrate how the method of parameter differentiation can be used to deduce the dependence of the critical Rayleigh number on a boundary constant, in this case, the Nusselt number. The method used here follows the approach used by Joseph [5] in a study of the parameter and domain dependence of eigenvalues for elliptic partial differential equations.

In the second section of this paper, the parameter differentiation technique is used to derive a geometric monotonicity result for the critical Rayleigh number for axisymmetric buoyancy-driven convection in a cylindrical container. The monotonicity condition is illustrated using the numerical results of Charlson and Sani [3]. In the third section of the paper, a geometric monotonicity result is derived for 3-dim. buoyancy-driven convection in a rectangular box, and this result is illustrated using the calculations of Catton [4]. Finally, in the fourth section of the paper, a monotonicity result is derived for the Nusselt number dependence of the critical Rayleigh number for axisymmetric convection in a cylindrical container with a free surface. This development is effectively a rederivation of an important result derived previously by Joseph and Shir [6]. The monotonicity condition for the Rayleigh number–Nusselt number dependence is illustrated using calculations presented by Vrentas *et al.* [7] for buoyancy-driven convection in a bounded cylindrical geometry.

2. DOMAIN DEPENDENCE IN A CYLINDRICAL GEOMETRY

In this paper, we shall consider buoyancy-driven convection in a Newtonian liquid layer contained in a container of finite size. All physical properties with the exception of density are constant, and viscous dissipation is assumed to be negligible. Furthermore, we introduce the usual Boussinesq approximation that the density in the body force term is a linear function of temperature. In all cases considered in this paper, we shall suppose that there are insulated rigid lateral walls and a constant temperature rigid bottom surface. In this section, a liquid layer of radius R and height l is contained in a cylindrical vessel. Two types of boundary conditions for the top surface are considered. In one case, the top surface is a rigid wall at constant temperature. In a second case, the top surface transfers heat, but not mass, to a constant temperature inviscid gas phase, and the heat exchange between gas and liquid is described using a constant heat transfer coefficient. Finally, we limit the analysis to axisymmetric velocity and temperature fields. For this problem, the origin of the cylindrical coordinate system is located at the top surface, and the unit vector along the z axis is parallel to the gravitational field. Furthermore, the dimensionless temperature is 0 at the bottom surface and 1 in the gas phase or at the top surface.

Under the above conditions, the dimensionless linear equations which govern the perturbations in stream function and temperature can be written as follows:

$$L\mathbf{w}_0 = 0. \tag{1}$$

Here, $\mathbf{w}_0$ is a two-component vector

$$\mathbf{w}_0 = \begin{bmatrix} \psi_0 \\ \theta_0 \end{bmatrix}, \tag{2}$$

L is the matrix operator

$$L = \begin{bmatrix} \dfrac{\beta^4 E^4}{r^2 (Ra)_0} & -\dfrac{1}{r}\dfrac{\partial}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial}{\partial r} & -\dfrac{(1+Nu)}{Nu}\nabla^2 \end{bmatrix}, \tag{3}$$

and E^2 is given by

$$E^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{4}$$

For a free surface with a constant heat transfer coefficient h, the boundary conditions for stream function and temperature perturbations can be written as follows:

$$\psi_0 = 0, \quad \frac{1}{r}\frac{\partial \psi_0}{\partial r} = \frac{\partial^2 \psi_0}{\partial r^2}, \quad r = 0, \quad 0 < z < \beta, \tag{5}$$

$$\psi_0 = 0, \quad \frac{\partial \psi_0}{\partial r} = 0, \quad r = 1, \quad 0 < z < \beta, \tag{6}$$

$$\psi_0 = 0, \quad \frac{\partial^2 \psi_0}{\partial z^2} = 0, \quad z = 0, \quad 0 < r < 1, \tag{7}$$

$$\psi_0 = 0, \quad \frac{\partial \psi_0}{\partial z} = 0, \quad z = \beta, \quad 0 < r < 1, \tag{8}$$

$$\frac{\partial \theta_0}{\partial r} = 0, \quad r = 0, \quad 0 < z < \beta, \tag{9}$$

$$\frac{\partial \theta_0}{\partial r} = 0, \quad r = 1, \quad 0 < z < \beta, \tag{10}$$

$$\frac{\partial \theta_0}{\partial z} = \frac{Nu}{\beta}\theta_0, \quad z = 0, \quad 0 < r < 1, \tag{11}$$

$$\theta_0 = 0, \quad z = \beta, \quad 0 < r < 1. \tag{12}$$

For a rigid, conducting top surface, equations (7) and (11) are replaced by the following equations:

$$\psi_0 = 0, \quad \frac{\partial \psi_0}{\partial z} = 0, \quad z = 0, \quad 0 < r < 1, \tag{13}$$

$$\theta_0 = 0, \quad z = 0, \quad 0 < r < 1. \tag{14}$$

The radius R is used as the reference length in forming the dimensionless distance variables, and κ/l is used as the reference velocity and l as the reference length in forming the dimensionless stream function.

In the above equations, the critical Rayleigh number is defined as

$$(Ra)_0 = \frac{l^3 \alpha \Delta T g}{\nu \kappa} \tag{15}$$

where, for the case of a free surface with a Robin boundary condition, ΔT is the critical temperature difference between the bottom surface and the gas phase. It is the usual practice to define a Rayleigh number, $(\overline{Ra})_0$, using the temperature difference across the liquid layer in the conductive state. The two Rayleigh numbers are related by the expression

$$(Ra)_0 = \frac{1+Nu}{Nu}(\overline{Ra})_0 \tag{16}$$

and, clearly, the difference in the definitions disappears for a conducting top surface where $Nu \to \infty$. Although either definition of the Rayleigh number can be utilized, we introduce the definition given by equation (15) here because we believe that it is generally more useful in arriving at a physical interpretation of stability results for this particular problem [7]. In particular, a finite value of $(\overline{Ra})_0$ for $Nu = 0$ is somewhat artificial since it requires an infinite temperature drop across the entire system.

For the above problem, it is convenient to define the following inner product of two real, vector-valued functions $\mathbf{a}$ and $\mathbf{b}$:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \int_0^\beta \int_0^1 \mathbf{a} \cdot \mathbf{b} \quad r \, dr \, dz. \tag{17}$$

Relative to this inner product, it can be shown in the usual way [3] that the system described by equation (1) and the corresponding boundary conditions is self-adjoint. To investigate the dependence of $(Ra)_0$ on β, we differentiate equation (1) and the associated boundary conditions with respect to β. This operation yields

$$L\mathbf{w}_a = \mathbf{K}, \tag{18}$$

$$\mathbf{w}_a = \begin{bmatrix} \dfrac{\partial \psi_0}{\partial \beta} \\ \dfrac{\partial \theta_0}{\partial \beta} \end{bmatrix}, \tag{19}$$

$$\mathbf{K} = \begin{bmatrix} \dfrac{1}{r}\dfrac{\partial \theta_0}{\partial r}\left\{\dfrac{\partial \ln (Ra)_0}{\partial \beta} - \dfrac{4}{\beta}\right\} + \dfrac{4\beta^3}{r^2 (Ra)_0}\left\{\dfrac{\partial^4 \psi_0}{\partial r^2 \partial z^2} - \dfrac{1}{r}\dfrac{\partial^3 \psi_0}{\partial r \partial z^2} + \dfrac{\partial^4 \psi_0}{\partial z^4}\right\} \\ -\dfrac{2(1+Nu)}{Nu\,\beta}\dfrac{\partial^2 \theta_0}{\partial z^2} \end{bmatrix} \tag{20}$$

and the vector $\mathbf{w}_a$ obeys the same boundary conditions as $\mathbf{w}_0$. The solvability condition for equation (18) takes the following form:

$$-\frac{\beta^4}{(Ra)_0}\frac{\partial[(Ra)_0 \beta^{-4}]}{\partial \beta}\int_0^\beta \int_0^1 \psi_0 \frac{\partial \theta_0}{\partial r} \, dr \, dz = \frac{4\beta^3}{(Ra)_0}\int_0^\beta \int_0^1 \frac{1}{r}\left(\frac{\partial^2 \psi_0}{\partial z^2}\right)^2 dr \, dz$$
$$+\frac{2(1+Nu)}{Nu\,\beta}\left\{\int_0^1 r\frac{Nu}{\beta}[\theta_0(r,0)]^2 \, dr + \int_0^\beta \int_0^1 r\left(\frac{\partial \theta_0}{\partial z}\right)^2 dr \, dz\right\} + \frac{4\beta^3}{(Ra)_0}\int_0^\beta \int_0^1 \frac{1}{r}\left(\frac{\partial^2 \psi_0}{\partial r \partial z}\right)^2 dr \, dz. \tag{21}$$

Since the RHS of this equation and the integral on the LHS are positive, we deduce the following inequality for the geometry dependence of $(Ra)_0$:

$$\frac{\partial[(Ra)_0 \beta^{-4}]}{\partial \beta} < 0. \tag{22}$$

Since $\beta = l/R = 1/\gamma$, where γ is the aspect ratio, equation (22) can also be written as

$$\frac{\partial[(Ra)_0 \gamma^4]}{\partial \gamma} > 0. \tag{23}$$

The above monotonicity condition is valid for each and every eigenvalue of the system and thus holds for the principal eigenvalue also.

This monotonicity result can be illustrated using the computations of Charlson and Sani [3] for axisymmetric convection in a rigid cylinder with conducting top and bottom surfaces and an insulating lateral wall. In Fig. 1, the lack of monotonic behavior for the $(Ra)_0$ vs β curve is evident as is the adherence of the numerical results to the condition stated in equation (22).

3. DOMAIN DEPENDENCE IN A RECTANGULAR GEOMETRY

In this section, we consider buoyancy-driven convection in a rectangular box with rigid walls, conducting top and bottom surfaces, and insulating lateral boundaries. For this problem, the origin of the coordinate system is located at the bottom surface, and z is the vertical direction. The unit vector in this direction is antiparallel to the gravitational field, and the dimensionless temperature is 0 at the top surface and 1 at the bottom surface. The dimensionless linear equations which describe the perturbations in the velocity and temperature fields can be written as

$$M\mathbf{Q} - \frac{1}{(Ra)_0}\mathbf{d}P_0 = 0 \tag{24}$$

$$\nabla \cdot \mathbf{v}_0 = 0 \tag{25}$$

where $\mathbf{Q}$ is a four-component vector

$$\mathbf{Q} = \begin{bmatrix} U_0 \\ V_0 \\ W_0 \\ \theta_0 \end{bmatrix}, \tag{26}$$

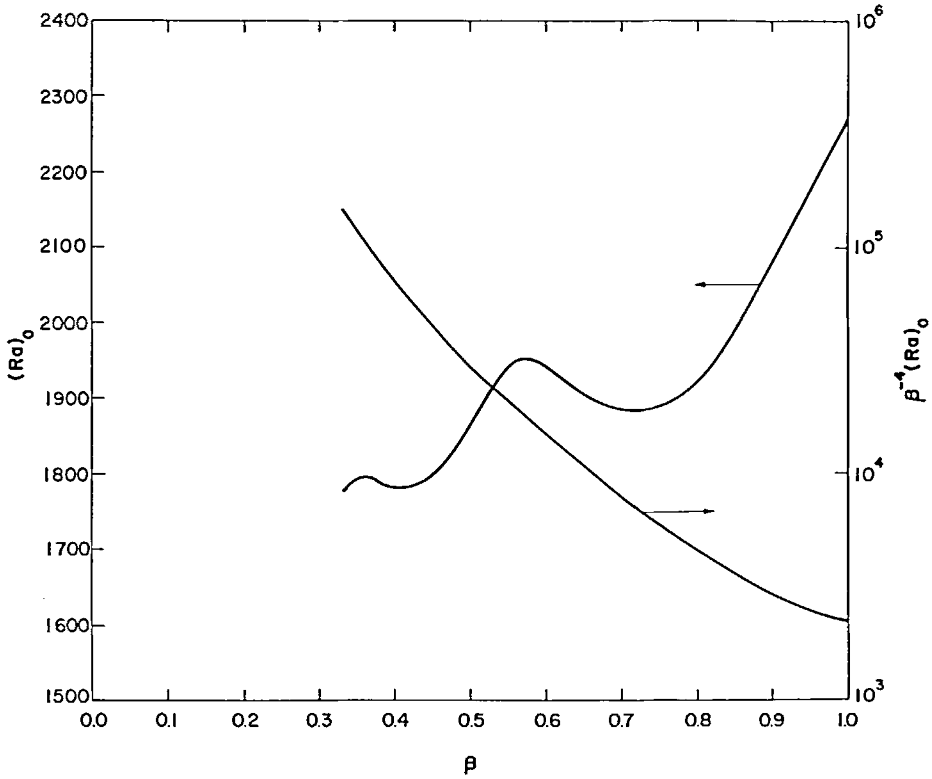

FIG. 1. Illustration of geometric monotonicity condition for cylindrical geometry using computations of Charlson and Sani [3].

d is a four-component gradient

$$\mathbf{d} = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \\ 0 \end{bmatrix}, \tag{27}$$

and M is the following matrix operator:

$$M = \begin{bmatrix} \dfrac{\nabla^2}{(Ra)_0} & 0 & 0 & 0 \\ 0 & \dfrac{\nabla^2}{(Ra)_0} & 0 & 0 \\ 0 & 0 & \dfrac{\nabla^2}{(Ra)_0} & 1 \\ 0 & 0 & 1 & \nabla^2 \end{bmatrix}. \tag{28}$$

Also, the boundary conditions for this problem can be written as follows:

$$\mathbf{v}_0 = 0 \quad \text{on} \quad z = 0, 1, \quad x = \pm L_1, \quad y = \pm L_2, \tag{29}$$

$$\theta_0 = 0 \quad \text{on} \quad z = 0, 1, \tag{30}$$

$$\frac{\partial \theta_0}{\partial x} = 0 \quad \text{on} \quad x = \pm L_1, \tag{31}$$

$$\frac{\partial \theta_0}{\partial y} = 0 \quad \text{on} \quad y = \pm L_2. \tag{32}$$

The height of the rectangular box is used as the reference length in forming the dimensionless distance variables, and the reference velocity is equal to κ divided by the height of the box.

We now study what effect changing the size of the box in the x direction has on the critical Rayleigh number. Differentiation of the above set of equations and boundary conditions with respect to L_1 gives

$$M\mathbf{Q}^* - \frac{1}{(Ra)_0}\mathbf{d}P_0^* = \mathbf{F}, \tag{33}$$

$$\nabla \cdot \mathbf{v}_0^* = \frac{1}{L_1}\frac{\partial U_0}{\partial x} \tag{34}$$

where $P_0^* = \partial P_0/\partial L_1$, $\mathbf{Q}^* = \partial Q/\partial L_1$, $\mathbf{v}_0^* = \partial \mathbf{v}_0/\partial L_1$, and $\mathbf{F}$ is the following four-component vector:

$$\mathbf{F} = \begin{bmatrix} \dfrac{2}{L_1(Ra)_0}\dfrac{\partial^2 U_0}{\partial x^2} - \dfrac{1}{L_1(Ra)_0}\dfrac{\partial P_0}{\partial x} \\ \dfrac{2}{L_1(Ra)_0}\dfrac{\partial^2 V_0}{\partial x^2} \\ -\dfrac{\theta_0}{(Ra)_0}\dfrac{\partial (Ra)_0}{\partial L_1} + \dfrac{2}{L_1(Ra)_0}\dfrac{\partial^2 W_0}{\partial x^2} \\ \dfrac{2}{L_1}\dfrac{\partial^2 \theta_0}{\partial x^2} \end{bmatrix}. \tag{35}$$

Since $\mathbf{Q}$ and $\mathbf{Q}^*$ clearly obey the same boundary conditions, it is easy to show in the usual way [3] that

$$\langle \mathbf{Q}^*, M\mathbf{Q}\rangle = \langle \mathbf{Q}, M\mathbf{Q}^*\rangle \tag{36}$$

where we have defined the following inner product of two real, vector-valued functions $\mathbf{a}$ and $\mathbf{b}$ over the volume V of the region:

$$\langle \mathbf{a},\mathbf{b}\rangle = \int_V \mathbf{a}\cdot\mathbf{b}\,\mathrm{d}V. \tag{37}$$

Also, it can be shown using equations (25) and (34) that

$$\langle \mathbf{Q}, \mathrm{d}P_0^*\rangle - \langle \mathbf{Q}^*, \mathrm{d}P_0\rangle = \int_V \frac{P_0}{L_1}\frac{\partial U_0}{\partial x}\,\mathrm{d}V. \tag{38}$$

By forming appropriate inner products from equations (24) and (33) and by using equations (36) and (38), it follows immediately that

$$0 = \langle \mathbf{Q},\mathbf{F}\rangle + \frac{\displaystyle\int_V \frac{P_0}{L_1}\frac{\partial U_0}{\partial x}\,\mathrm{d}V}{(Ra)_0}. \tag{39}$$

This result can also be written as

$$\begin{aligned} 0 = &-\frac{2}{L_1(Ra)_0}\left\{\int_V\left[\left(\frac{\partial U_0}{\partial x}\right)^2 + \left(\frac{\partial V_0}{\partial x}\right)^2 + \left(\frac{\partial W_0}{\partial x}\right)^2\right]\mathrm{d}V\right\} \\ &-\frac{2}{L_1(Ra)_0}\int_V U_0\frac{\partial P_0}{\partial x}\,\mathrm{d}V - \frac{2}{L_1}\int_V\left(\frac{\partial \theta_0}{\partial x}\right)^2\mathrm{d}V \\ &-\frac{\partial \ln(Ra)_0}{\partial L_1}\int_V \theta_0 W_0\,\mathrm{d}V \end{aligned} \tag{40}$$

and, by further utilization of equation (24), we finally arrive at

$$\frac{\partial[(Ra)_0 L_1^4]}{\partial L_1} = \frac{N}{D}, \tag{41}$$

$$\begin{aligned} N = &-2L_1^3\int_V\left[\left(\frac{\partial U_0}{\partial y}\right)^2 + \left(\frac{\partial U_0}{\partial z}\right)^2 + \left(\frac{\partial V_0}{\partial y}\right)^2 + \left(\frac{\partial V_0}{\partial z}\right)^2 + \left(\frac{\partial W_0}{\partial y}\right)^2 + \left(\frac{\partial W_0}{\partial z}\right)^2\right]\mathrm{d}V \\ &-2L_1^3(Ra)_0\int_V\left[\left(\frac{\partial \theta_0}{\partial y}\right)^2 + \left(\frac{\partial \theta_0}{\partial z}\right)^2\right]\mathrm{d}V \\ &-2L_1^3\int_V [\nabla U_0\cdot\nabla U_0]\,\mathrm{d}V, \end{aligned} \tag{42}$$

$$D = \int_V \theta_0\nabla^2\theta_0\,\mathrm{d}V. \tag{43}$$

Since all integrals in both N and D are negative, it follows that

$$\frac{\partial[(Ra)_0 L_1^4]}{\partial L_1} > 0. \tag{44}$$

An equivalent partial derivative for a change of the size of the box in the y direction can of course also be formulated. Although the monotonicity conditions for the axisymmetric and fully 3-dim. problems are effectively the same, there are obvious differences in the methods of derivation.

This particular monotonicity result can be illustrated using the calculations of Catton [4] for convection in a rigid rectangular region with perfectly conducting top and bottom surfaces and insulating vertical walls. It should be noted that Catton found critical Rayleigh numbers which were based on the existence of finite rolls in the rectangular region. Davies-Jones [8] has shown that finite rolls which are aligned perpendicular to insulating, rigid sidewalls are not an exact solution to the linearized equations. We shall suppose here that the values of $(Ra)_0$ reported by Catton are sufficiently accurate for the purpose of illustrating the monotonicity result. The absence of monotonic behavior for the $(Ra)_0$ vs L_1 curve is illustrated in Fig. 2. It is also clear that the computed results adhere to the monotonicity condition given by equation (44). Interpolation of the numerical results of Catton to form the $(Ra)_0$ vs L_1 curve presented in Fig. 2 was carried out using the above monotonicity result. It is evident that meaningful interpolation of the numerical data for this case is made possible only by utilization of the inequality presented in equation (44).

4. NUSSELT NUMBER DEPENDENCE

In the final section of this paper, we study the Nusselt number dependence of $(Ra)_0$ for the axisymmetric buoyancy-driven convection problem considered

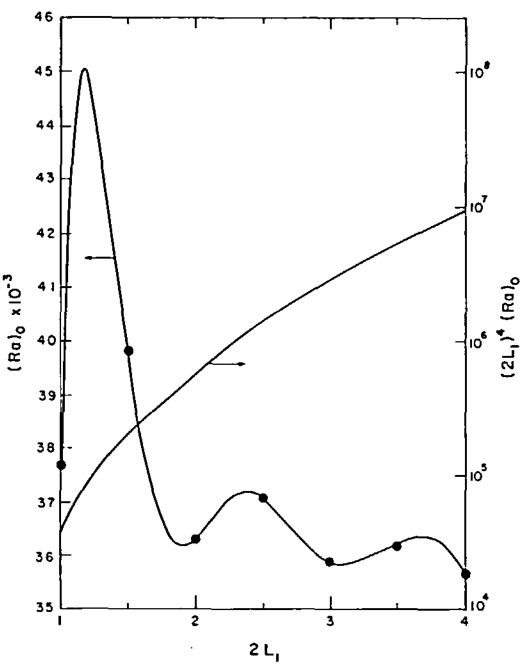

FIG. 2. Illustration of geometric monotonicity condition for rectangular geometry using computations of Catton [4] with $2L_2 = 0.125$. Solid circles represent reported values of the critical Rayleigh number.

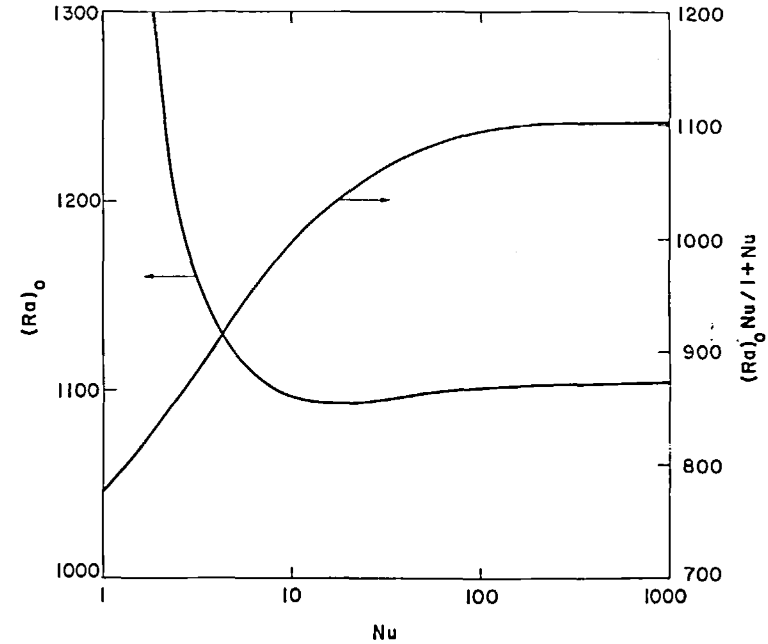

FIG. 3. Illustration of monotonicity condition for Nusselt number dependence for cylindrical geometry using computations of Vrentas *et al.* [7] with $R/l = 8$.

above. The dependence of $(Ra)_0$ on Nu can be examined by differentiating equation (1) and the corresponding boundary conditions with respect to Nu. This procedure leads to the system

$$L\mathbf{w}_b = \mathbf{N}, \tag{45}$$

$$\mathbf{w}_b = \begin{bmatrix} \dfrac{\partial \psi_0}{\partial Nu} \\ \dfrac{\partial \theta_0}{\partial Nu} \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}, \tag{46}$$

$$\mathbf{N} = \begin{bmatrix} \dfrac{1}{r(Ra)_0} \dfrac{\partial \theta_0}{\partial r} \dfrac{\partial (Ra)_0}{\partial Nu} \\ -\dfrac{1}{r} \dfrac{\partial \psi_0}{\partial r} \dfrac{(1+Nu)}{Nu} \dfrac{\mathrm{d}}{\mathrm{d}\,Nu}\left(\dfrac{Nu}{1+Nu}\right) \end{bmatrix}, \tag{47}$$

and the vector $\mathbf{w}_b$ has the same boundary conditions as $\mathbf{w}_0$ with the exception of the temperature condition at $z = 0$ which becomes

$$\frac{\partial B}{\partial z} - \frac{NuB}{\beta} = \frac{\theta_0}{\beta}. \tag{48}$$

Now, if we form the following inner product expression:

$$\langle \mathbf{w}_0, L\mathbf{w}_b \rangle - \langle \mathbf{w}_b, L\mathbf{w}_0 \rangle = \langle \mathbf{w}_0, \mathbf{N} \rangle \tag{49}$$

we can derive the result

$$\frac{\partial \left[\dfrac{(Ra)_0 Nu}{1+Nu} \right]}{\partial Nu} = \frac{(Ra)_0}{\beta} \frac{\displaystyle\int_0^1 r[\theta_0(r,0)]^2 \,\mathrm{d}r}{\displaystyle\int_0^\beta \int_0^1 \psi_0 \frac{\partial \theta_0}{\partial r} \,\mathrm{d}r\,\mathrm{d}z} > 0. \tag{50}$$

It is evident that the monotonicity condition applies to $(\overline{Ra})_0$, and it also follows from equation (50) that $\partial(Ra)_0/\partial Nu$ need not have the same sign for all Nu. As $Nu \to 0$, a uniform temperature exists in the liquid layer in the conductive state, and a large temperature difference [large $(Ra)_0$] is necessary for cellular convection. Although there will be a general decrease of $(Ra)_0$ as Nu increases away from $Nu = 0$, it appears that a minimum in the $(Ra)_0$ vs Nu curve cannot be excluded. Indeed, calculations carried out by Vrentas *et al.* [7] for this particular geometry with $R/l = 8$ show that there is a minimum in the $(Ra)_0$ vs Nu curve near $Nu = 10$, as is illustrated in Fig. 3. This figure also shows adherence to the monotonicity condition given by equation (50).

Joseph and Shir [6] used an energy method to deduce that the stability limit increases monotonically with the Nusselt number. Since there are no subcritical instabilities for the type of convection problem considered here, the conclusion of Joseph and Shir is valid for linear theory, and equation (50) is thus equivalent to their monotonicity condition. We have simply presented an alternative derivation based on parameter differentiation and the theory of differential operators. Finally, we note that exact calculations based on the linear equations for fluid layers of infinite horizontal extent [9, 10] appear to represent the first instance where it is shown that $(\overline{Ra})_0$ increases monotonically with increasing Nu.

REFERENCES

1. S. Ostrach and D. Pnueli, The thermal instability of completely confined fluids inside some particular configurations, *J. Heat Transfer* 85, 346–354 (1963).

2. P. A. Jennings and R. L. Sani, Some remarks on thermoconvective instability in completely confined regions, *J. Heat Transfer* **94**, 234–236 (1972).
3. G. S. Charlson and R. L. Sani, Thermoconvective instability in a bounded cylindrical fluid layer, *Int. J. Heat Mass Transfer* **13**, 1479–1496 (1970).
4. I. Catton, The effect of insulating vertical walls on the onset of motion in a fluid heated from below, *Int. J. Heat Mass Transfer* **15**, 665–672 (1972).
5. D. D. Joseph, Parameter and domain dependence of eigenvalues of elliptic partial differential equations, *Arch. Rational Mech. Anal.* **24**, 325–351 (1967).
6. D. D. Joseph and C. C. Shir, Subcritical convective instability Part 1. Fluid layers, *J. Fluid Mech.* **26**, 753–768 (1966).
7. J. S. Vrentas, R. Narayanan and S. S. Agrawal, Free surface convection in a bounded cylindrical geometry, *Int. J. Heat Mass Transfer* **24**, 1513–1529 (1981).
8. R. P. Davies-Jones, Thermal convection in an infinite channel with no-slip sidewalls, *J. Fluid Mech.* **44**, 695–704 (1970).
9. E. M. Sparrow, R. J. Goldstein and V. K. Jonsson, Thermal instability in a horizontal fluid layer: effect of boundary conditions and non-linear temperature profile, *J. Fluid Mech.* **18**, 513–528 (1964).
10. D. A. Nield, Surface tension and buoyancy effects in cellular convection, *J. Fluid Mech.* **19**, 341–352 (1964).

DEPENDANCE DES NOMBRES DE RAYLEIGH CRITIQUES VIS-A-VIS DU DOMAINE

Résumé—On étudie la dépendance des nombres de Rayleigh critiques pour la convection naturelle vis-à-vis de la taille du domaine. La méthode de différenciation du paramètre est utilisée afin d'établir des conditions de monotonie pour la dépendance nombre de Rayleigh-géométrie dans le cas de la convection axisymétrique dans un réservoir cylindrique et dans le cas de la convection tridimensionnelle dans une boite rectangulaire. On étudie aussi la dépendance du nombre de Rayleigh critique vis-à-vis du nombre de Nusselt.

DIE ABHÄNGIGKEIT DER KRITISCHEN RAYLEIGH-ZAHL VON DER RAUMGRÖSSE

Zusammenfassung—Die Abhängigkeit der kritischen Rayleigh-Zahl bei freier Konvektion von der Größe des betrachteten Raumes wird untersucht. Um Monotonitätsbedingungen für die Abhängigkeit der Rayleigh-Zahl von der Geometrie bei achsensymmetrischer Konvektion in einem zylindrischen Behälter und bei dreidimensionaler Konvektion in einem rechteckigen Behälter herzustellen, wird die Methode der Parameter-Differentiation angewandt. Die Abhängigkeit der kritischen Rayleigh-Zahl von der Nusselt-Zahl wird ebenfalls untersucht.

ВЛИЯНИЕ РАЗМЕРОВ ИССЛЕДУЕМОЙ ОБЛАСТИ НА КРИТИЧЕСКИЕ ЗНАЧЕНИЯ ЧИСЛА РЭЛЕЯ

Аннотация—Исследуется зависимость критических значений числа Рэлея от размеров области при конвекции за счет подъемной силы. С целью определения условий монотонности для зависимости числа Рэлея от геометрии при осесимметричной конвекции в цилиндре и при трехмерной конвекции в прямоугольном объеме используется метод параметрического дифференцирования. Исследуется также зависимость критического числа Рэлея от числа Нуссельта.